5.3: Approx. of Sobolev functions by smooth functions

Mollifiers: Smooth a rough enough Sobolev functions by averaging, i.e. convolving with a "smooth bump function."

Def$^{\underline{n}}$: 1) Consider the standard mollifier $\eta(x) \in C_c^\infty(\mathbb{R}^n)$, defined by

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1. \end{cases}$$

where $C>0$ s.t. $\int_{\mathbb{R}^n} \eta \, dx = 1$. Note: $\eta(x) \geq 0 \ \forall x \in \mathbb{R}^n$ and $\operatorname{supp}(\eta) = \overline{\mathbb{B}_1(0)}$.

2) Given $\varepsilon>0$, define $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. One can check that

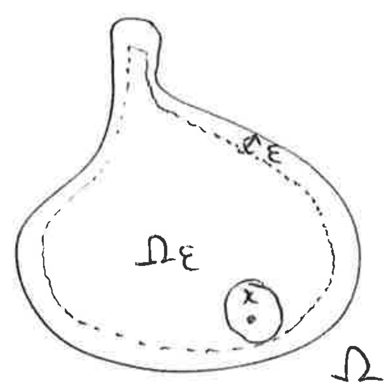
- $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$; $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$; $\operatorname{supp}(\eta_\varepsilon) = \overline{\mathbb{B}_\varepsilon(0)}$
- As $\varepsilon \downarrow 0$, $\varepsilon^{-n}\uparrow$, so η_ε looks tall and concentrated, yet $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$.

3) Given $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$, define its mollification / regularisation of u at scale $\varepsilon>0$ as

$$\begin{aligned} u^\varepsilon(x) &= (\eta_\varepsilon * u)(x) \\ &= \int_\Omega \eta_\varepsilon(x-y)u(y)\,dy \\ &= \int_{\mathbb{B}_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,dy \quad \forall x \in \Omega_\varepsilon. \end{aligned}$$

- u^ε is defined only on Ω_ε so that we have "ε-room" to average u.
- Essentially, u^ε is a weighted, smooth average of u over $\mathbb{B}_\varepsilon(x)$.

Here, $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.
$= \{x \in \Omega : \overline{\mathbb{B}_\varepsilon(x)} \subset \Omega\}$.



Remark: The precise formula for η doesn't really matter, as long as $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int \eta = 1$. Sometimes it is nice to have rotational symmetries on η.

Lemma: (Mollification commutes with weak ~~differention~~ differentiation).

Let $\Omega \subset \mathbb{R}^n$ open, α a multi-index, and $u \in L^1_{loc}(\Omega)$ s.t. weak derivative $D^\alpha u$ exists. Then $\forall x \in \Omega_\varepsilon$ we have that

$$D^\alpha(u^\varepsilon) = \underbrace{D^\alpha}_{\text{classical derivative}}(\underbrace{\eta_\varepsilon * u}_{\text{smooth function}}) = \eta_\varepsilon * D^\alpha u = (\underbrace{D^\alpha u}_{\text{weak derivative}})^\varepsilon$$

smooth function.

Proof: $D^\alpha u^\varepsilon(x) = \int_\Omega D^\alpha_x [\eta_\varepsilon(x-y)] u(y) dy$

$$= (-1)^{|\alpha|} \int_\Omega D^\alpha_y [\eta_\varepsilon(x-y)] u(y) dy$$

$$= (-1)^{|\alpha|+|\alpha|} \int_\Omega \eta_\varepsilon(x-y) D^\alpha u(y) dy \quad [\text{def}^{\underline{n}} \text{ of weak-derivative}]$$

$$= (\eta_\varepsilon * D^\alpha u)(x). \quad \blacksquare$$

[Allows us to relate weak derivatives with partial derivatives in the classical sense].

Corollary: Let $\Omega \subset \mathbb{R}^n$ open, connected and assume $u \in L^1_{loc}(\Omega)$. If $Du = 0$ a.e., then $u = 0$ a.e.

Proof: Consider u^ε, then $(Du)^\varepsilon = Du^\varepsilon = 0 \Rightarrow u^\varepsilon$ must be constant on each connected component of Ω_ε.

Let $\tilde{u}(x) \overset{a.e.}{=} \lim_{\varepsilon \to 0} u_\varepsilon(x)$. Then we know that $\tilde{u}(x)$ is constant function on Ω. Also, $\tilde{u}(x) = u(x)$ a.e. on Ω. $\blacksquare$

"Averages of L^1_{loc} functions converge pointwise a.e. to the function itself as $r \downarrow 0$".

"Expect a function and its mollification to be close..."

Thm: (~~the~~ Lebesgue Differentiation Thm).

If $u \in L^1_{loc}(\mathbb{R}^n)$, then (1) $u(x_0) = \lim_{r \downarrow 0} \fint_{B(x_0,r)} u(x) dx$ for almost every point $x_0 \in \mathbb{R}^n$.

(Stronger). (2) $\lim_{r \downarrow 0} \fint_{B(x_0,r)} |u(x) - u(x_0)| dx = 0$ for almost every point $x_0 \in \mathbb{R}^n$.

A point $x_0 \in \mathbb{R}^n$ in which (2) holds is called a Lebesgue point of f.

<u>Thm</u>: Thm 7, pg 714 + Thm 1, Section 5.3, pg 264.

Suppose $\Omega \subset \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$.

(1) $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$

2) $u^\varepsilon \to u$ pointwise ae in Ω as $\varepsilon \to 0^+$ (true $\forall$ Lebesgue points).

3) If $u \in C(\Omega)$, then $u^\varepsilon \to u$ uniformly on compact subsets of Ω.

Moreover, for any $K \subset\subset \Omega$ and $\varepsilon > 0$ suff. small,

4) (a) If $u \in L^p_{loc}(\Omega)$ with $p \in [1,\infty]$, then $\|u^\varepsilon\|_{L^p(K)} \leq \|u\|_{L^p(\Omega)}$

(b) If $u \in L^p_{loc}(\Omega)$ with $p \in [1,\infty)$, then $u^\varepsilon \to u$ in $L^p_{loc}(\Omega)$

5) (a) If $u \in W^{k,p}(\Omega)$ with $p \in [1,\infty]$, then $\|u^\varepsilon\|_{W^{k,p}(K)} \leq \|u\|_{W^{k,p}(\Omega)}$

(b) If $u \in W^{k,p}(\Omega)$ with $p \in [1,\infty)$, then $u^\varepsilon \to u$ in $W^{k,p}_{loc}(\Omega)$.

<u>Proof</u>: 1) Easy, just need to check differentiation under integral sign.
(Well, DCT or note that difference quotient $\to$ partial derivatives uniformly).

2) Suppose x is a Lebesgue point, then

$$|u^\varepsilon(x) - u(x)| = \left| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)[u(y)-u(x)]\,dy \right|$$

$$\leq \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u(y)-u(x)|\,dy$$

$$\leq \frac{C\cancel{\varepsilon^n}}{\cancel{\varepsilon^n}} \|\eta_\varepsilon\|_{L^\infty} ⨍_{B(x,\varepsilon)} |u(y)-u(x)|\,dy \to 0 \text{ as } \varepsilon \to 0.$$

3) Since u is continuous, all $x \in \Omega$ are Lebesgue points, so $u^\varepsilon \to u$ at each point.
Suppose $u^\varepsilon \in C^\infty(V)$ for some $K \subset\subset \Omega$. Since u^ε spills out from V, we consider an intermediate domain $K \subset\subset \Omega'' \subset\subset \Omega$, so that $\text{supp}(u^\varepsilon) \subset \Omega''$ for ε suff. small. Since u is uniformly continuous on Ω'', the estimate above holds uniformly $\forall x \in K$, which means that $u^\varepsilon \to u$ uniformly on K.

4) (a) Immediate if $p=\infty$: $|u^\varepsilon(x)| = \left| \int_{\mathbb{B}_\varepsilon(x)} \eta_\varepsilon(x-y)\, u(y)\,dy \right|$

$$\leq \|\eta_\varepsilon\|_{L^1(\Omega)} \|u\|_{L^\infty(\Omega)}.$$

Suppose $p \in [1,\infty)$, then.

$$|u^\varepsilon(x)| \leq \int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u(y)|\,dy$$

$$= \int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)^{\frac{1}{p}} \eta_\varepsilon(x-y)^{1-\frac{1}{p}} |u(y)|\,dy$$

$$\cancel{\leq \| \eta_\varepsilon * u \|_{L^p(\mathbb{B}(x,\varepsilon))}}$$

$$\leq \left(\int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u(y)|^p\,dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)\,dy \right)^{1-\frac{1}{p}}.$$

$$= \left(\int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u(y)|^p\,dy \right)^{\frac{1}{p}}$$

Thus, for any $K \subset\subset \Omega$, we integrate $|u^\varepsilon(x)|^p$ over K and get

$$\int_K |u^\varepsilon(x)|^p\,dx \leq \int_K \left(\int_{\mathbb{B}(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u(y)|^p\,dy \right) dx$$

$$= \int_K \left(\int_{\mathbb{B}(0,\varepsilon)} \eta_\varepsilon(z)\,|u(x-z)|^p\,dz \right) dx$$

$$= \int_{\mathbb{B}(0,\varepsilon)} \eta_\varepsilon(z) \left(\int_K |u(x-z)|^p\,dx \right) dz$$

$$\leq \|u\|^p_{L^p(\Omega)} \quad \text{for } \varepsilon \text{ small enough.}$$

~~(b) WTS~~

4)(b) WTS: $\|u-u^\varepsilon\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$, $\forall K \subset\subset \Omega$.

Consider $K \subset\subset \Omega'' \subset\subset \Omega$, where ~~$u^\varepsilon$ is 0~~ $\operatorname{supp}(u^\varepsilon) \subset \Omega''$ for suff small $\varepsilon > 0$. Since $u \in L^p_{loc}(\Omega'')$, $\exists$ $v \in C(\Omega'')$ s.t. $\|u-v\|_{L^p(\Omega'')} < \delta$.

We can mollify v to get v^ε and so $\|v-v^\varepsilon\|_{L^p(\Omega)} < \delta$.

$$\Rightarrow \|u-u^\varepsilon\|_{L^p(K)} \leq \|u-v\|_{L^p(K)} + \|v-v^\varepsilon\|_{L^p(K)} + \|v^\varepsilon-u^\varepsilon\|_{L^p(K)}$$
$$\leq \|u-v\|_{L^p(\Omega'')} + \|v-v^\varepsilon\|_{L^p(\Omega)} + \|v-u\|_{L^p(\Omega'')}$$
$$< 3\delta.$$

5)(a) This is easy now. For $p \in [1,\infty)$,

$$\|u^\varepsilon\|^p_{W^{k,p}(K)} = \sum_{|\alpha|\leq k} \|D^\alpha(u^\varepsilon)\|^p_{L^p(K)}$$
$$= \sum_{|\alpha|\leq k} \|(D^\alpha u)^\varepsilon\|^p_{L^p(K)}$$
$$\leq \sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)}$$

(b) Similarly, $$\|u^\varepsilon - u\|^p_{W^{k,p}(K)} = \sum_{|\alpha|\leq k} \|D^\alpha(u^\varepsilon - u)\|^p_{L^p(K)}$$
$$= \sum_{|\alpha|\leq k} \|(D^\alpha u)^\varepsilon - (D^\alpha u)\|^p_{L^p(K)}$$
$$\to 0 \text{ as } \varepsilon \to 0.$$

$\blacksquare$

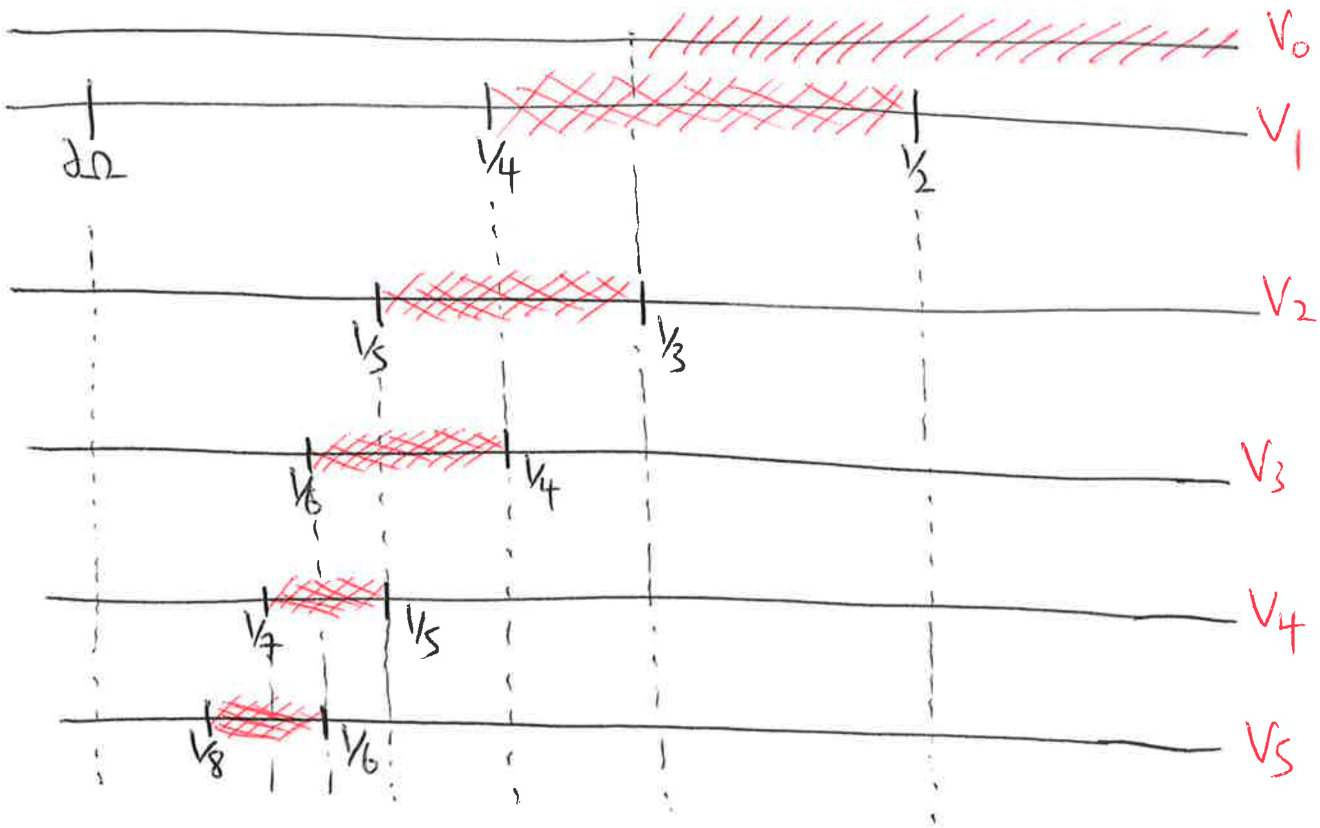
Thm: Thm 2, pg 265.

Suppose $\Omega \subset \mathbb{R}^n$ is open and $u \in W^{k,p}(\Omega)$ for $p \in [1,\infty)$. Then $\exists \{u_m\}_{m=1}^{\infty} \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$ s.t. $u_m \to u$ in $W^{k,p}(\Omega)$.

Proof: 1) Fix $\delta > 0$. Consider the locally finite open cover of Ω, $\bigcup_{j=1}^{\infty} V_j = \Omega$.

$$V_0 = \{x \in \Omega : d(x, \partial\Omega) > \tfrac{1}{3}\}$$

$$V_j = \{x \in \Omega : \tfrac{1}{j+3} < d(x,\partial\Omega) < \tfrac{1}{j+1}\}, \; j = 2, 3, \ldots$$



Note that any point x in V_j is ~~not~~ being covered by only finitely many covers.

Let $\{\xi_j\}_{j=1}^{\infty}$ be a smooth partition of unity subordinate to the open covers $\{V_j\}_{j=1}^{\infty}$, that is,
$$\xi_j \in [0,1], \; \xi_j \in C_c^{\infty}(V_j), \; \sum_{j=1}^{\infty} \xi_j = 1 \text{ on } \Omega.$$

For any $u \in W^{k,p}(\Omega)$, $\xi_j u \in W^{k,p}(\Omega)$; moreover, $\mathrm{supp}(\xi_j u) \subset V_j$.

By construction, $u = \sum_{j=1}^{\infty} \xi_j u$, which is now a sum of $C_c^{\infty}(\Omega)$ functions.

2) Mollify each $(\xi_j u)$ to get $u^j := \eta_{\varepsilon_j} * (\xi_j u)$. ~~Since mollification commutes with weak differentiation,~~ We know that $u^j \to \xi_j u$ in $W^{k,p}_{loc}(\Omega)$, but since ξ_j has compact support, $u^j \to \xi_j u$ in $W^{k,p}(\Omega)$. Thus for each $j = 1, 2, \dots$, we can find ε_j small enough such that. $\left(0 < \varepsilon_j < \frac{1}{2j+1} \right)$.

$$\| u^j - \xi_j u \|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^j}$$

[ε_j must be getting smaller and smaller so that u^j doesn't "spill out" from Ω].

3) Define the function $V = \sum_{j=1}^{\infty} u^j = \sum_{j=1}^{\infty} \eta_{\varepsilon_j} * (\xi_j u)$. This series may not even converge in $W^{k,p}$. ~~But since any open set $K \subset\subset \Omega$ intersects finitely many V_j's, $V|_K$ only contains finitely many nonzero terms.~~

~~$\Rightarrow$ V is pointwise convergent and $V \in C^\infty(\Omega)$.~~

But $V \in C^\infty(\Omega)$ since each $x \in \Omega$ is in the support of only finitely many u^j. $\nabla\nabla$ not in $C_c^\infty(\Omega)$, though.

Finally, since $u = \sum_{j=1}^{\infty} \xi_j u$, for each $K \subset\subset \Omega$ we have.

$$\| V - u \|_{W^{k,p}(K)} \le \sum_{j=1}^{N_K} \| u^j - \xi_j u \|_{W^{k,p}(\Omega)} \le \sum_{j=1}^{N_K} \frac{\delta}{2^j} \le \delta.$$

Taking the supremum over sets $K \subset\subset \Omega$, we conclude that $\| V - u \|_{W^{k,p}(\Omega)} \le \delta$.

This shows that C^∞ function is dense on $W^{k,p}(\Omega)$. $\blacksquare$